Recovery of systems with a linear filter and nonlinear delay feedback in periodic regimes

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We propose a set of methods for the estimation of the parameters of time-delay systems with a linear filter and nonlinear delay feedback performing periodic oscillations. The methods are based on an analysis of the system response to regular external perturbations and are valid only for systems whose dynamics can be perturbed. The efficiency of the methods is illustrated using both numerical and experimental data.

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I. INTRODUCTION

Self-sustained oscillators with delay-induced dynamics are widespread in nature and engineering. Their abundance results from such fundamental features as the finite velocity of signal propagation and delayed feedback inherent to many physical [1–3], chemical [4,5], climatic [6,7], and biological [8–10] systems and processes. In recent years the problem of estimation of the parameters of time-delay systems from experimental time series has attracted a lot of attention. The solution of this problem allows one to predict the system behavior under parameter variation, to test the adequacy of a model, to classify the systems and their regimes, and to recover the parameters that cannot be directly measured in an experiment. A variety of methods have been proposed to reconstruct the model equations of time-delay systems from their chaotic time series [11–19]. However, these methods fail if a time-delay system executes a periodic motion [20]. In this case one cannot recover the delay time because it is not possible to define from periodic oscillations whether the system is governed by a delay-differential equation or an ordinary differential equation. But in practice many important systems with delay-induced dynamics operate in periodic or nearly periodic regimes [21–23]. To estimate their parameters one has to develop new techniques. The problem of recovering the parameters of time-delay systems in periodic regimes was addressed recently in Ref. [24], where for the delay time estimation it was proposed to disturb the system by a short-correlated noisy signal of large amplitude and to analyze the correlation function. The delay was identified by a jump in the second derivative of the correlation function, which is a manifestation of the system response after the delay time to the short-correlated strong disturbances.

In this paper we propose another approach for reconstructing the parameters of time-delay systems performing periodic oscillations based on an analysis of the system response to regular external impulsive perturbations. The use of a periodic impulsive disturbance can be preferable in some cases. If the external disturbance is strong enough, it leads to the appearance of a transient process. As a result, the system motion takes place in a wider region of phase space. It gives us additional information about the system dynamics and can be useful for the recovery of system parameters. Besides the delay time, the method allows one to reconstruct other essential parameters of the system. The approach based on the analysis of the transient processes is studied in Sec. II. The proposed method is applied to the time series of a model system and experimental data gained from an electronic oscillator with delayed feedback. However, in a number of cases a strong disturbance of the system is undesirable because it can result in an essential change of the system behavior or even destruction of the system. In these cases the use of small disturbances is preferable. In Sec. III we propose a method of parameter estimation for time-delay systems in periodic regimes based on an analysis of the system response to a small periodic disturbance. Since the noise amplitude can be greater than the amplitude of small external perturbations, we use the method of accumulation [25] for analyzing the system response. The method is verified by using it for the recovery of time-delay systems of different order. In Sec. IV we summarize our results.

II. USING TRANSIENT PROCESSES FOR TIME-DELAY SYSTEM RECOVERY

Let us consider a special time-delay system described by the first-order delay-differential equation

$$\varepsilon_1 \dot{x}(t) = -x(t) + f(x(t-\tau_1)), \qquad (1)$$

where τ_1 is the delay time, function f defines nonlocal correlations in time, and the parameter ε_1 characterizes the inertial properties of the system. In Ref. [16] it was found that the time series of time-delay systems governed by Eq. (1) practically have no extrema separated in time by the delay time. If the system (1) performs chaotic oscillations, the extrema in its time series are located irregularly and the time intervals between these extrema can take different values. Defining, for different values of τ, the number N of situations when the points of the time series separated in time by τ are both extremal, we can construct the $N(\tau)$ plot and recover the delay time τ_1 as the value at which the absolute minimum of $N(\tau)$ is observed [16].

However, if the system (1) performs periodic oscillations, this technique fails because the extrema in the time series are located regularly. As a result, the $N(\tau)$ plot exhibits several peaks separated by intervals where $N=0$. Figure 1 shows an example of such a time series and the corresponding $N(\tau)$ plot for the system (1) with $\tau_1=300$, $\varepsilon_1=10$, $f(x)=\lambda-x^2$, and $\lambda=1$. For a given parameter of nonlinearity λ, the system (1) shows periodic oscillations of period $T_a=619$ [Fig. 1(a)].

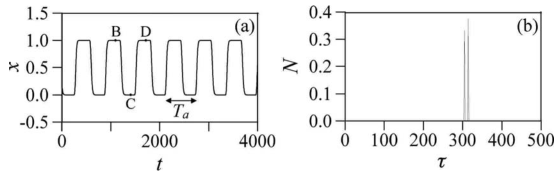

FIG. 1. (a) The time series of periodic self-sustained oscillations of the system (1). (b) Number N of pairs of extrema in the time series separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series.

Since the temporal realization of $x(t)$ is asymmetric, the $N(\tau)$ plot [Fig. 1(b)] displays two peaks at $\tau=305$ and $\tau=314$, which correspond to the distances between the maximum B and the minimum C and between the minimum C and the maximum D, respectively. Hence, the dependence of N on τ cannot be used for reconstructing the delay time from periodic time series.

Let us disturb the system (1) by an external signal $F(t)$. The disturbed system is given by

$$\varepsilon_1 \dot{x}(t) = -x(t) + f(x(t-\tau_1)) + F(t). \tag{2}$$

We consider the external signal $F(t)$ having the form of rectangular pulses with amplitude A, period T, and duration M. If the disturbance is strong enough, it leads to the onset of a transient process that gives us additional information about the system dynamics and can help in recovering the system parameters [26]. In particular, the appearance of a large number of additional extrema in the time series during the transients results in the occurrence of a pronounced minimum in the $N(\tau)$ plot at the delay time.

Figure 2 shows the time series and the dependence $N(\tau)$ for system (2) calculated at the same parameter values as in system (1) and the pulse signal parameters $A=0.5$, $T=490$, and $M=0.2T$. The time series in Fig. 2(a) looks like a chaotic one and contains a large number of irregularly located extrema. The $N(\tau)$ plot, constructed with a step of τ variation equal to 1, exhibits the minimum at the true value of the delay time $\tau=\tau_1=300$ [Fig. 2(b)]. The location of the absolute maximum of $N(\tau)$ is defined by the parameter ε_1 [16].

The proposed method of delay time estimation is efficient in a wide range of external signal parameters. For $A=0.5$ and $M=0.2T$ the period T of pulses arbitrarily taken from $T=1.2\tau_1$ to $T=1.8\tau_1$ ensures an accurate recovery of τ_1 for the considered values of the system parameters. There are also narrower intervals of T less than τ_1 and greater than $2\tau_1$ providing an accurate reconstruction of the delay time.

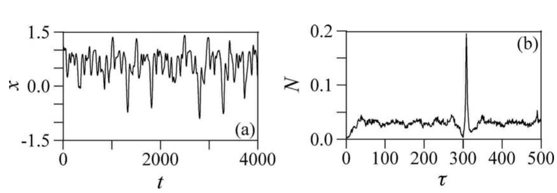

FIG. 2. (a) The time series of the perturbed system (2). (b) Number N of pairs of extrema in the time series separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series. $N_{min}(\tau)=N(300)$.

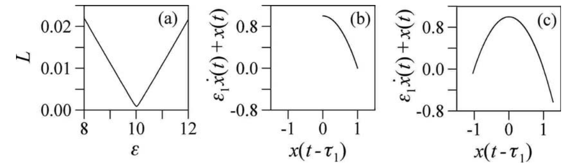

FIG. 3. (a) Length L of a line connecting all points ordered with respect to $x(t-\tau_1)$ in the $(x(t-\tau_1), \varepsilon\dot{x}(t)+x(t))$ plane of the system (1), as a function of ε. $L(\varepsilon)$ is normalized to the number of points in the plane. $L_{min}(\varepsilon)=L(10.00)$. (b) The nonlinear function recovered from the periodic time series of the system (1). (c) The nonlinear function recovered from the time series of the perturbed system (2).

Knowledge of oscillation mode of undisturbed system (1) can help in choosing T. One should take into account that for the most typical principal mode the delay time τ_1 is always less than $T_a/2$ regardless of the nonlinear function of system (1). For the higher-order modes, which can take place for very small ε_1, we have $\tau_1 < nT_a/2$, where n is the mode order. Furthermore, the range of values of T ensuring an accurate recovery of τ_1 can be different for different nonlinear functions. For a first approximation we suggest to take T close to $0.8T_a$ and then vary it, if necessary. As a criterion for a successful choice of T and other parameters of the pulse signal one can use the presence of a single pronounced minimum located just before the absolute maximum in the $N(\tau)$ plot [see Fig. 2(b)].

We have also tested the method by varying the pulse duration from $M=0.05T$ to $M=0.5T$. The method is still efficient but needs the amplitude A to increase for small M. With an increase of M, the value of A can be decreased. The considered impulsive disturbance can have an advantage over the system disturbance by a strong stochastic force used in Ref. [24] for the delay time estimation in periodic regime. The system is disturbed now not permanently but by a pulse signal. It is easy to control the parameters of the impulsive disturbance, choosing an appropriate variant from short but strong disturbance to long but low-amplitude one.

To recover the parameter ε_1 and the nonlinear function f from the system (1) periodic time series one can use the method proposed in Ref. [11] and modified in Ref. [27], where it was applied to chaotic time series of the time-delay system. Following this method, we have to project the system (1) trajectory on the plane $(x(t-\tau_1), \varepsilon\dot{x}(t)+x(t))$ under variation of ε and calculate the length $L(\varepsilon)$ of a line, connecting all points ordered with respect to the abscissa in the mentioned plane. When the parameter ε coincides with the true parameter ε_1, the points of the projection lie on a single-valued curve, reproducing the nonlinear function f. The length $L(\varepsilon)$ is minimal in this case. In the case of inaccurate parameter estimation a set of points in the plane, to which the trajectory of the system is projected, becomes more dispersed. As a result, the polygon line connecting these points has a greater length than in the previous case.

Figure 3(a) shows the $L(\varepsilon)$ plot constructed at the recovered delay time $\tau_1=300$ and the step of ε variation equal to 0.01. The time derivative $\dot{x}(t)$ is estimated from the time series by applying a local parabolic approximation. The minimum of $L(\varepsilon)$ is observed at $\varepsilon=\varepsilon_1=10.00$. The nonlinear function, reconstructed for $\tau_1=300$ and $\varepsilon_1=10$ from the sys-

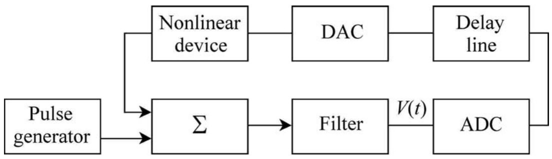

FIG. 4. Block diagram of the electronic oscillator with delayed feedback disturbed by a pulse signal.

tem (1) periodic time series, is presented in Fig. 3(b). Such a technique allows us to recover only a fragment of the function f, since the oscillations take place only in a small region of phase space because of their periodicity. For a more extended recovery of the nonlinear function one should exploit the time series of the perturbed system (2). In this case only the points of the time series corresponding to the intrinsic dynamics of the time-delay system should be used for plotting $\varepsilon_1\dot{x}(t)+x(t)$ versus $x(t-\tau_1)$. It means that we have to exploit the points from the intervals between the successive pulses of the external signal. The nonlinear function recovered in this manner is depicted in Fig. 3(c). It coincides well with the true quadratic function of the system (1).

To test the method efficiency in the presence of noise we apply it to the data produced by adding a zero-mean Gaussian white noise to the time series of Eq. (2). For the cases where the additive noise has a standard deviation of up to 10% of the standard deviation of the data without noise the $N(\tau)$ plot still shows the minimum accurately at the delay time. As for the parameter ε_1 and the nonlinear function, they are recovered with a good accuracy using the time series of the perturbed system—i.e., exploiting only the points from the intervals between the successive external pulses.

The proposed method can be applied for reconstruction of time-delay systems of high order and multiple delays in periodic regimes. In these cases for estimating the parameters of the disturbed system one can use the methods developed in Ref. [28] for reconstruction of high-order time-delay systems and systems with several delays from chaotic time series. However, the method is not valid for such time-delay systems as those considered in Ref. [29], containing the nonlinear term, which is a function of the nondelayed signal.

Let us consider application of the method to experimental time series gained from an electronic oscillator with delayed feedback perturbed by an external pulse signal. A block diagram of the experimental setup is shown in Fig. 4. The delay of the signal $V(t)$ for time τ_1 is provided by a delay line constructed using digital elements. The role of the nonlinear device is played by an amplifier constructed using bipolar transistors and having a quadratic transfer function. The inertial properties of the oscillator are defined by a low-frequency first-order RC filter, which resistance R and capacitance C specify $\varepsilon_1=RC$. The analog and digital elements of the scheme are connected with the help of analog-to-digital and digital-to-analog converters. The signal from a pulse generator is applied to the oscillator using the summator Σ. The considered oscillator is governed by the first-order delay-differential equation

$$RC\dot{V}(t) = -V(t) + f(V(t-\tau_1)) + F(t), \qquad (3)$$

where $V(t)$ and $V(t-\tau_1)$ are the delay line input and output voltages, respectively. In the absence of an impulsive disturbance the oscillator shows at $\tau_1=4.16$ ms and $RC=0.32$ ms periodic self-sustained oscillations of period $T_a=8.88$ ms.

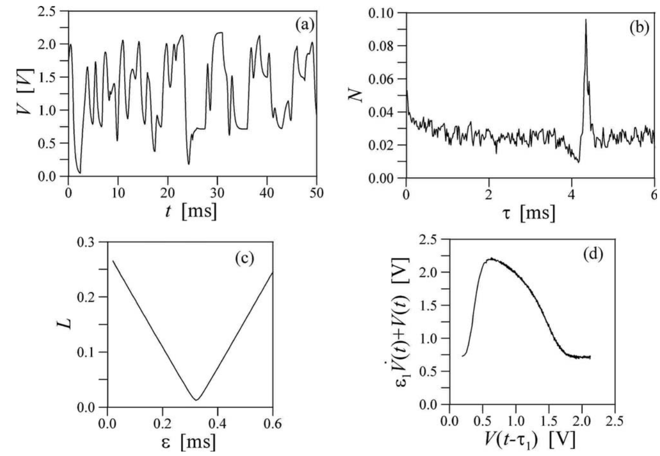

FIG. 5. (a) Experimental time series of the electronic oscillator with delayed feedback under impulsive disturbance. (b) Number N of pairs of extrema in the time series separated in time by τ, as a function of τ. $N(\tau)$ is normalized to the total number of extrema in the time series. $N_{min}(\tau)=N(4.16\text{ ms})$. (c) The dependence $L(\varepsilon)$ normalized to the number of points. $L_{min}(\varepsilon)=L(0.32\text{ ms})$. (d) The recovered nonlinear function.

Using an analog-to-digital converter with sampling frequency $f_s=50$ kHz we record the signal $V(t)$ at the pulse signal parameters $A=1.6$ V, $T=7.5$ ms, and $M=1.5$ ms. Part of the experimental time series is shown in Fig. 5(a). For various τ values we count the number N of situations when $\dot{V}(t)$ and $\dot{V}(t-\tau)$ are simultaneously equal to zero and construct the $N(\tau)$ plot [Fig. 5(b)]. The step of τ variation in Fig. 5(b) is equal to the sampling time $T_s=0.02$ ms. The absolute minimum of $N(\tau)$ takes place exactly at the delay time $\tau=\tau_1=4.16$ ms. The $L(\varepsilon)$ plot, constructed with the recovered $\tau_1=4.16$ ms and the step of ε variation equal to 0.01 ms, exhibits the minimum at $\varepsilon=\varepsilon_1=0.32$ ms [Fig. 5(c)]. In Fig. 5(d) the nonlinear function recovered from the time series of the disturbed system is presented. This function coincides closely with the true transfer function f of the amplifier.

III. CASE OF TIME-DELAY SYSTEM DISTURBANCE BY A SMALL PERIODIC SIGNAL

The use of strong disturbance leading to the appearance of a transient process in a time-delay system performing periodic oscillations is not always possible. Because of the peculiarities of the system dynamics, the strong disturbance can result in undesirable qualitative change of the system behavior or even cause a destruction of the system. In these cases it is preferable to use small disturbances for estimating the system parameters. In this section we propose an original method for the recovery of time-delay systems in periodic regimes based on the analysis of the system response to perturbation by a small periodic signal.

Let us consider a ring time-delayed feedback system composed of a delay line, nonlinear device, and filter [Fig. 6(a)], performing periodic self-sustained oscillations $x(t)$ with pe-

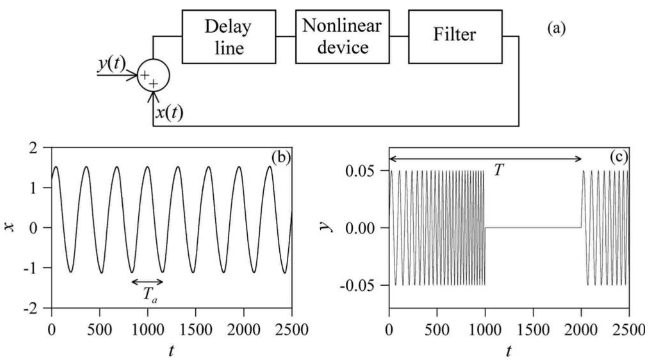

FIG. 6. (a) Block diagram of the ring time-delayed feedback system under external perturbation. (b) The time series of periodic self-sustained oscillations. (c) The time series of the external periodic radio pulses with linearly increasing filling frequency.

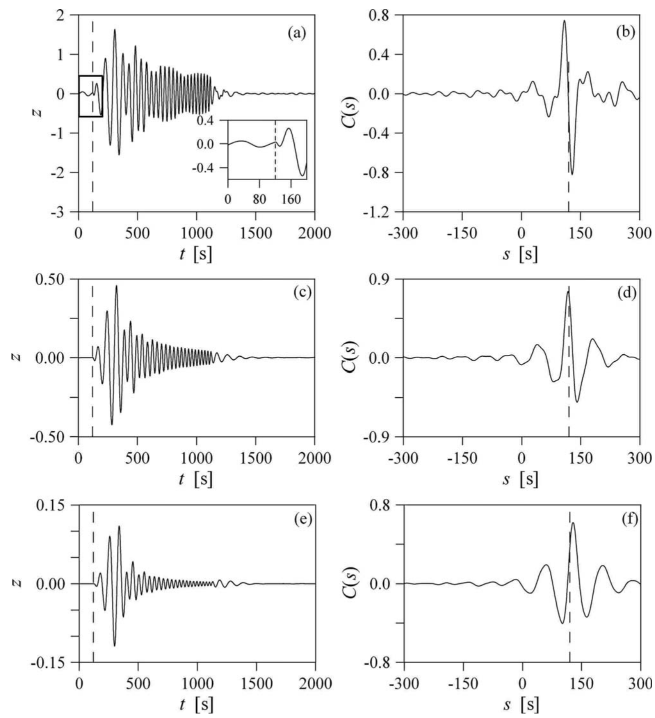

FIG. 7. Superposition z of 100 responses of the time-delay system to the small periodic impulsive disturbance and the cross-correlation function $C(s)$ of the signals of disturbance and the system response for the system of the first [(a) and (b)], second [(c) and (d)], and third [(e) and (f)] order. The dashed lines indicate the time corresponding to the delay time $\tau_1=120$ s. The inset in (a) is the enlarged fragment of $z(t)$ in the vicinity of $t=120$ s.

riod T_a [Fig. 6(b)]. We disturb the system by an external signal $y(t)$ having the form of rectangular radio pulses with linearly increasing filling frequency [Fig. 6(c)]. The filling period is always less than the period of oscillations, T_a. The pulse period T greatly exceeds the preliminary estimation of the delay time, which is usually less than $T_a/2$. The form of the model equation for this system is determined by the filter parameters and the point of the external signal injection into the ring time-delay system. In the case where the filter is composed of three identical in-series low-frequency RC filters and the signal $y(t)$ is added to the system between the filter and the delay line [Fig. 6(a)], the considered system is governed by the third-order delay-differential equation

$$\varepsilon_3\frac{d^3x(t)}{dt^3}+\varepsilon_2\frac{d^2x(t)}{dt^2}+\varepsilon_1\frac{dx(t)}{dt} = -x(t)+f(x(t-\tau_1)+y(t-\tau_1)), \quad (4)$$

where $\varepsilon_1=3RC$, $\varepsilon_2=3(RC)^2$, and $\varepsilon_3=(RC)^3$. If the filter is a two-section RC filter with identical sections, then $\varepsilon_3=0$, $\varepsilon_2=(RC)^2$, and $\varepsilon_1=2RC$. In the case of a simple low-frequency RC filter the model equation takes the form of a first-order delay-differential equation. As a nonlinear function we choose the sigmoid

$$f(x)=\frac{c}{1+a\exp[-b(x-x_0)]}-\frac{c}{1+a\exp[b(x-x_0)]}, \quad (5)$$

providing stable periodic oscillations in a wide region of the control parameter variation. Note that the form of the nonlinear function is not critical for the method application. Instead of the function (5) one can choose, for example, the quadratic function used in Sec. II.

The system was numerically investigated at a delay time $\tau_1=120$ s, filter cutoff frequency $f_F=1/RC\approx0.0032$ Hz, and nonlinear function parameters $a=1$, $b=2$, $c=-2$, and $x_0=0.5$. The radio pulses had an amplitude $A=0.05$, period $T=2000$ s, duration $M=1000$ s, and filling frequency f_r linearly increasing from 0.01 Hz to 0.04 Hz. The self-sustained oscillations had an amplitude of about 1 and the period T_a taking the values of 320 s, 410 s, and 510 s for the system of the first, second, and third order, respectively.

To separate the intrinsic dynamics of the system and its response to perturbation we filter the signals $x(t)$ and $y(t)$ with a high-frequency high-order filter with cutoff frequency $f_c=0.01$ Hz, which is much greater than the basic frequency of the oscillations, $f_a=1/T_a$, but is not higher than the filling frequency f_r of the radio pulses. Then we divide the time series of the filtered self-sustained oscillations into parts beginning at the time moments of the appearance of the radio pulses and sum these parts [Fig. 7(a)]. Such a technique based on the method of accumulation [25] allows one to increase significantly the amplitude of the system response to perturbations, because the summation of the pulse signals passed through the system feedback circuit is carried out in the same phase. On the contrary, the intrinsic oscillations of the time-delay system are summed in different phases compensating each other. As a result, it becomes possible to extract the system response to small disturbance even in the presence of noise. The influence of noise tends to zero with increasing number of superimposed parts of noisy time series since the noisy signal has a random amplitude of different sign. It should be noted that the method of accumulation [25] has been known for a long time, but it was applied mainly in the field of the transmission and detection of radio signals in the presence of noise. In this paper we apply this method for estimating the parameters of time-delay systems.

At the next step we calculate the cross-correlation function of the signals of disturbance $y(t)$ and the system response $z(t)$:

$$C(s) = \frac{\langle y(t)z(t+s)\rangle}{\sqrt{\langle y(t)^2\rangle\langle z(t)^2\rangle}}, \quad (6)$$

where the angular brackets denote averaging over time. We use this function for estimating the delay time and the order of the filter.

Let us consider what happens with the signal $y(t)$ as it passes through the ring time-delay system [Fig. 6(a)]. At first, the signal is retarded by the delay line. The cross-correlation function of $y(t)$ and $y(t-\tau_1)$ exhibits the maximum shifted by τ_1 with respect to zero. Second, the signal $y(t-\tau_1)$ is transformed by the nonlinear device. Since the nonlinear function (5) has a negative slope, the nonlinear device changes the signal phase by π [30]. As a result, the cross-correlation function of $y(t)$ and $f(y(t-\tau_1))$ instead of the maximum has the minimum at the delay time. At last, the signal $f(y(t-\tau_1))$ passes through the filter, which changes the signal phase. As the frequency f_r is several times greater than the filter cutoff frequency f_F, the phase shift is almost constant for the entire signal. For a first-order filter, the change of signal phase is about $-\pi/2$. As a consequence, the cross-correlation function (6) of $y(t)$ and the system response $z(t)$ takes zero value at the delay time. In this case τ_1 can be estimated as the value at which the first zero of $C(s)$, located to the left of the deepest minimum of $C(s)$, is observed. The plot of $C(s)$ in Fig. 7(b) gives the estimation of $\tau_1=119$ s.

For a second-order filter, the change of the phase of the considered signal is close to $-\pi$. Hence, the delay time corresponds to the location of the maximum of the function (6). The delay time estimated from Fig. 7(d) is $\tau_1=119$ s. A third-order filter shifts the signal phase by $-3\pi/2$. In this case the location of the first zero of $C(s)$, observed to the left of the main maximum, gives the estimation of τ_1. The delay time recovered from Fig. 7(f) is $\tau_1=118$ s. Note that a high-frequency filter with cutoff frequency f_c does not change the phase relation between the signals of perturbation and the system response because both these signals pass through the same filter.

The delay time and the order of the filter of the delayed feedback system can be also estimated analyzing alone the system response to small disturbance. The time-delay system response to small periodic disturbance, accumulated using a superposition of 100 responses, is presented in Figs. 7(a), 7(c), and 7(e) for the system of the first, second, and third order, respectively. The delay of the response gives us the following estimation: $\tau_1=121$ s for Fig. 7(a), $\tau_1=122$ s for Fig. 7(c), and $\tau_1=123$ s for Fig. 7(e). For the first-order time-delay system we can see oscillations of z in Fig. 7(a) in the range of times from 0 until the delay time, impeding the recovery of τ_1. In this case we estimate the delay time from the inset of Fig. 7(a) as the value at which $z(t)$ exhibits a break. For more clearness we plot the second derivative of $z(t)$ in Fig. 8. The sharp minimum of $d^2z(t)/dt^2$ (see the inset in Fig. 8) is observed at $t=121$ s giving the same estimation of τ_1 as Fig. 7(a).

As can be seen from Fig. 7, the amplitude of the response signal $z(t)$ decreases with the increase of the radio pulse filling frequency. Such behavior of the response signal is defined by the amplitude-frequency characteristic of the filter. Outside of a filter transmission band the attenuation of the signal grows at the rate of about $6n$ dB/octave, where n is the order of the filter [31]. We obtain the following numerical estimation of the response amplitude decreasing: 7.5 dB/octave for Fig. 7(a), 13.5 dB/octave for Fig. 7(c), and 16.8 dB/octave for Fig. 7(e). These results allow us to define with confidence the order of the filter and the model equation.

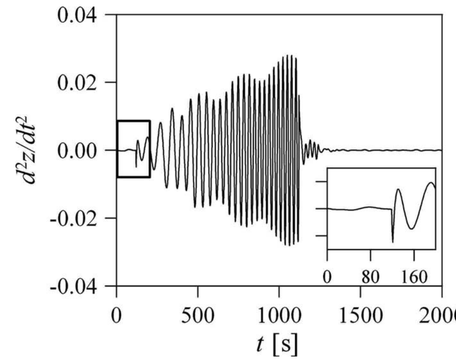

FIG. 8. The second derivative of the response signal $z(t)$ depicted in Fig. 7(a) for the first-order time-delay system. The inset is an enlarged fragment of $d^2z(t)/dt^2$ at small t values.

The method is verified in the presence of noise. Figure 9 shows the results of the method application to the time series of the considered first-order time-delay system corrupted by additive zero-mean Gaussian white noise. In Figs. 9(a) and 9(b) the standard deviation of noise is 5% of the standard deviation of data without noise (the signal-to-noise ratio is about 26 dB) and about 200% of the standard deviation of the disturbance. The parameters of the time-delay system and the disturbance are chosen the same as those in Figs. 7(a) and 7(b). The system response [Fig. 9(a)] gives an estimation of $\tau_1=121$ s for the delay time. The estimation obtained from

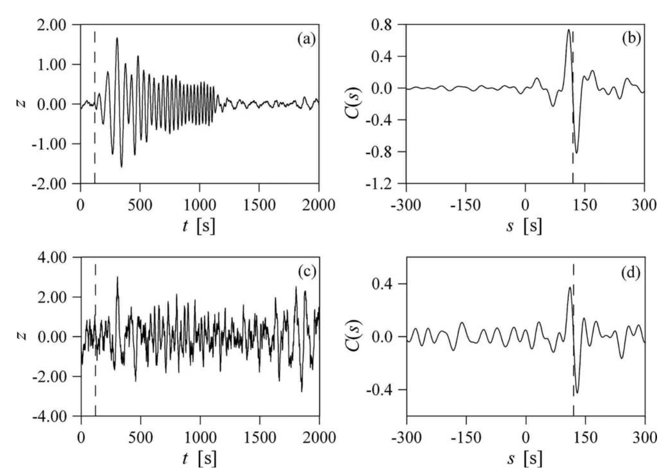

FIG. 9. Superposition z of 100 responses of the first-order time-delay system to the small periodic impulsive disturbance and the cross-correlation function $C(s)$ of the signals of disturbance and the system response in the presence of additive Gaussian white noise for noise level of 5% [(a) and (b)] and 100% [(c) and (d)]. The dashed lines indicate the time corresponding to the delay time $\tau_1=120$ s.

the cross-correlation function [Fig. 9(b)] is $\tau_1 = 119$ s. In spite of the noise presence, the delay time is recovered with good accuracy. The decrease of the response amplitude, estimated from Fig. 9(a) as 7.0 dB/octave, indicates that the system is of first order. In Figs. 9(c) and 9(d) the standard deviation of noise is equal to the standard deviation of data without noise and is 40 times greater than the standard deviation of the disturbance. For such a high level of noise the system response [Fig. 9(c)] fails to estimate the delay time. However, the cross-correlation function still gives a close estimation of τ_1. The delay time estimated from Fig. 9(d) is $\tau_1 = 119$ s.

IV. CONCLUSION

We have proposed a set of methods for the reconstruction of stable systems with a linear filter and nonlinear time-delay feedback performing periodic oscillations. The first method is based on a system perturbation by a strong regular external signal giving rise to a transient process. The use of a periodic impulsive signal with easily controlled parameters allows one to vary in a wide range the strength and duration of the disturbance, choosing an appropriate variant for the system under investigation. The method uses the statistical analysis of time intervals between extrema in the time series of the disturbed system and the projection of infinite-dimensional phase space of the time-delay system to suitably chosen low-dimensional subspaces. It is an extension of the methods developed in Refs. [11,15,16,28] for reconstruction of time-delay systems from chaotic time series to the case of periodic regimes. The proposed method allows one to recover not only the delay times as the other methods developed for time-delay systems in periodic regimes, but also the nonlinear functions and the parameters characterizing the inertial properties of the system. This method is successively applied to the time series of a model delay-differential equation and an experimental time series acquired from an electronic oscillator with delayed feedback, disturbed by an external signal.

The second method is based on an analysis of the time-delay system response to a weak periodic disturbance. It is an attempt to estimate the parameters of a time-delay system in a periodic regime that does not exploit a significant change of the system dynamics. For example, the amplitude of the external impulsive signal was only about 3% of the amplitude of the periodic oscillations of the time-delay system considered in the paper. The method exploits an investigation of the cross-correlation function of the signals of perturbation and the system response. For extracting the response of the system to small periodic signals of disturbance we used the method of accumulation. It is shown that the proposed method is efficient for the estimation of the delay time and the order of delay-differential equation of the ring time-delay system. We verified the method by applying it to the time series of model delay-differential equations of different order, including those heavily corrupted by noise. The delay time was recovered with good accuracy even in the presence of noise whose amplitude was comparable with the amplitude of the time-delay system oscillations.

Periodic behavior is typical for some time-delay systems [21–23] in spite of their potential ability to exhibit a high-dimensional chaotic dynamics. The proposed methods can be useful for experimental investigation of such systems. However, it should be emphasized that both methods developed in the paper can be applied only in the case when we are able to perturb the system dynamics. One cannot use these methods for the recovery of time-delay systems having at one's disposal only unperturbed periodic time series.

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